MCWP Senior High Mathematics League

Contest 1 **October 17, 2005**

Individual Questions

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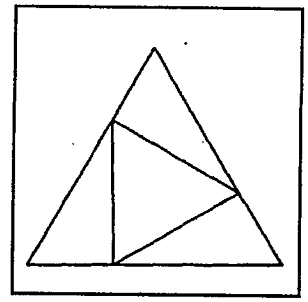
INDIVIDUAL ROUND 1 — 15 MINUTES

1. Two consecutive price reductions of the same percent reduced the price of an item from \$25.00 to \$16.00. By what percent was the price reduced each time?

2. What is the area of the smallest circle in which a pair of perpendicular chords can have lengths of 6 and 8?

3. In a large equilateral triangle is inscribed a smaller one, its vertices at trisection points of the sides of the larger, as shown. If the perimeter of the larger triangle is 9, what is the perimeter of the smaller equilateral triangle ?



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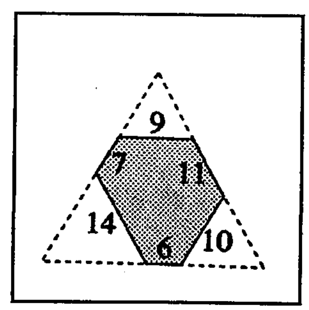
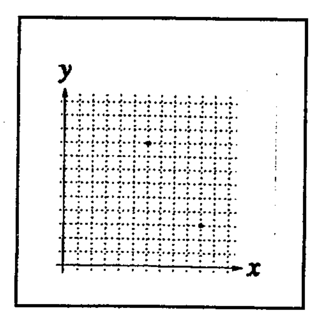
INDIVIDUAL ROUND 2 — 15 MINUTES

4. Mary's area code is a positive 3-digit number. Add 7 to it and the result is divisible by 7. Add 8 instead, and the result is divisible by 8. Add 9 instead, and the result is divisible by 9. What is Mary's area code?

5. The diagram shows that an equiangular hexagon with side-lengths, 6, 7, 9, 10, 11, and 14 can be inscribed in an equilateral triangle with side-length 30. This same equiangular hexagon can also be inscribed in an equilateral triangle with side-length n, where $n \neq 30$. What is this value of n ?



6. The ordered pairs, $(6,9)$ and $(10,3)$ are the coordinates of two opposite vertices of a square, as shown. What are the coordinates of the other two vertices ?



MCWP Senior High Mathematics League

Contest #1 **October 17, 2005**

TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

T-1. The entries in a 7 X 7 magic square array of numbers are the integers 1 through 49, inclusive. The sum of the entries in every row, and in both major diagonals is the same, and is called the "magic sum". What is the value of this "magic sum"?

T-2. What are all real values of x which satisfy the equation,

$$\frac{\sqrt{21+x}+\sqrt{21-x}}{\sqrt{21+x}-\sqrt{21-x}}=\frac{21}{x}\ ?$$

T-3. How many of the positive integers less than 10,000 contain the digit "1" at least once?

T-4. The lengths of the sides of a right triangle are in the ratio $3:4:5$. If the length of one of the three altitudes of this triangle is 60, what is the greatest possible area of this triangle?

T-5. The roots of $x^2 - 26x + c = 0$ are r and s. If $19r + 94s = 1994$, what is the value of c ?

MCWP Senior High Mathematics League

Contest #2 **INDIVIDUAL QUESTIONS** **November 14, 2005**

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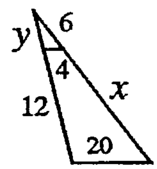
INDIVIDUAL ROUND 1 — 15 MINUTES

1. What is the positive value of x which satisfies $x = \dfrac{1}{1+\dfrac{1}{1+x}}$?

2. The hour and minute hands of an accurate clock overlap at noon. What is the least positive value of k for which these two hands will again overlap in k hours?

3. As shown, a triangle of base-length 4 is cut from a triangle of base-length 20 so the bases are parallel. If the segments shown have the lengths indicated, what is the value of $x-y$?



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INDIVIDUAL ROUND 2 — 15 MINUTES

4. Eanie, Meanie, Miney, and Mo found some coins. Eanie took half the coins and 2 more; Meanie took half the remaining coins and 2 more; Miney took half the remaining coins and 2 more; Mo took the 1 remaining coin. How many coins did they find?

5. If $x^2 + x - 5 = 0$, what is the value of $x^4 + 2x^3 + x^2 - 5$?

6. The only pendant I sell can be made from silver, gold, or platinum. I sell 10 silver, 4 gold, and 1 platinum pendant for \$1690. I sell 7 silver, 3 gold, and 1 platinum pendant for \$1260. At these prices, for how many dollars do I sell 1 silver, 1 gold, and 1 platinum pendant?

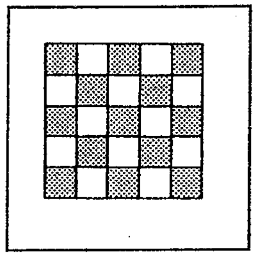
MCWP Senior High Mathematics League

Contest #2 November 14, 2005

TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

T–1. What is the largest number of checkers that could be placed on the 5 X 5 grid shown at the right, so there is at most 1 checker on any small square, and there are at most 4 checkers in any column, in any row, and in any diagonal?



T–2. If Lynn collected \$10 selling 17¢ and 23¢ items, what is the least number of 17¢ items Lynn could have sold?

T–3. Single copies of a book cost \$16 each, but purchasers of 20 or more books pay only \$13 per book. What are all values of $n < 20$ for which one could buy 20 books at a lower total cost than one could buy exactly n books?

T–4. What are all real numbers x which satisfy: $\log_2\left(\dfrac{x-1}{x+1}\right) > 1$?

T–5. In an arithmetic progression, the sum of the third and fifth terms is 14 and the sum of the first 12 terms is 129. If the nth term is 193, what is the value of n ?

MCWP Senior High Mathematics League

Contest #3 **INDIVIDUAL QUESTIONS** **December 12, 2005**

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INDIVIDUAL ROUND 1 — 15 MINUTES

1. Club dues are 5¢ the 1st week, 10¢ the 2nd week, and so on, increasing by 5¢ each week. How much money does a full-time club member pay for dues during a 36-week club year?

2. Yogi Berra, when asked if he wanted his pizza cut into 4 slices or 8 slices, said "Four. I don't think I can eat 8." With 1 straight cut, you can slice a circle into 2 pieces. A 2nd cut that crosses the 1st will create 4 pieces, and a 3rd cut can produce as many as 7 pieces. At most how many pieces can you get when you slice a circle with 6 straight cuts?

3. Give me 2 nickels, and I'll have 3 times as many nickels as dimes. Give me 3 dimes instead, and I'll have twice as many nickels as dimes. What's the total value of my nickels and dimes?

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INDIVIDUAL ROUND 2 — 15 MINUTES

4. A crow started at point A and flew 2 km due south, then 7 km due west, then 7 km due south, then 33 km due west, ending at point B. If $AB = x$ km, what is the value of x?

5. What are all real values of x which satisfy $|1-2x| > 1+3x$?

6. If $x > 0$ and $y > 0$ satisfy $x^2 + xy = 7$ and $y^2 + xy = 11$, what is the value of $x + y$?

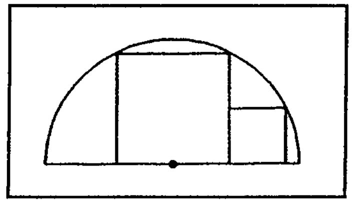
MCWP Senior High Mathematics League

Contest #3 **December 12, 2005**

TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

T–1. Two squares are inscribed in a semicircle as shown. If the area of the smaller square is 25, what is the area of the larger square?



T–2. Brian has 3 pennies, 3 nickels, 2 dimes, and 2 quarters. How many different sums of money can he make using one or more ot these 11 coins?

T–3. Pat divided x by y and got a quotient of 3 and a remainder of 7. Pat then divided x by 2y and got a quotient of q and a remainder of r. If x, y, q *and* r are all positive integers, what is the least possible value of r ?

T–4. What are all real numbers x for which $\dfrac{3^{\sqrt{12x}} + 3}{4} = 3^{\sqrt{3x}}$?

T–5. If N is an even integer between 1 and 2005, what is the probability that N^2 is divisible by 8 ?

MCWP Senior High Mathematics League

Contest #4 **INDIVIDUAL QUESTIONS** **January 9, 2006**

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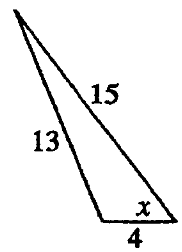
INDIVIDUAL ROUND 1 — 15 MINUTES

1. If the dividend is 6 times the divisor, and the divisor is 6 times the quotient, what is the value of the dividend?

2. Convex pentagon P's smallest angle measures 38°. Excluding P's largest and smallest angles, the degree-measure of each of P's other angles is the average of the degree-measures of the two angles nearest to it in size. What is the degree-measure of P's largest angle?

3. In the 4-13-15 triangle pictured, what is the value of $\tan x$?



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INDIVIDUAL ROUND 2 — 15 MINUTES

4. Of the two values of x which satisfy $3^{2x^2-7x+3} = 4^{x^2-x-6}$, one is rational. What is this value?

5. The real-valued function f is defined by $f(x) = x^2+7x+k$ and $f(k) = -16$. What is the value of $f(6)$?

6. Write the value of $\sqrt{5+\sqrt{21}} - \sqrt{5-\sqrt{21}}$ in simplest form.

MCWP Senior High Mathematics League

Contest #5 **INDIVIDUAL QUESTIONS** **February 6, 2006**

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INDIVIDUAL ROUND 1 — 15 MINUTES

1. How long is a diagonal of the square whose area and perimeter are numerically equal?

2. Let $f(x) = x + \frac{1}{x}$. If $y = f(a)$, write the value of $(f(a))^3 - f(a^3)$ in simplest form in terms of y.

3. What is the only positive number x which satisfies $(2x)^{2006} = 5^{1003}x^{1003}$?

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INDIVIDUAL ROUND 2 — 15 MINUTES

4. The bisector of $\angle A$ of $\triangle ABC$ intersects the circumcircle of $\triangle ABC$ at E, and $\overline{AE}$ intersects $\overline{BC}$ at D. If $AB = 20$, $AC = 12$, and $DE = 8$, what is AD?

5. What are both ordered pairs of consecutive integers (x,y) which satisfy $5x - 4y = 46$?

6. Determine the sum of the infinite series $\frac{1}{5} + \frac{2}{25} + \frac{1}{125} + \frac{2}{625} + \ldots$, if the numerators alternate between 1 and 2, as indicated, and the nth denominator is 5^n.

MCWP Senior High Mathematics League

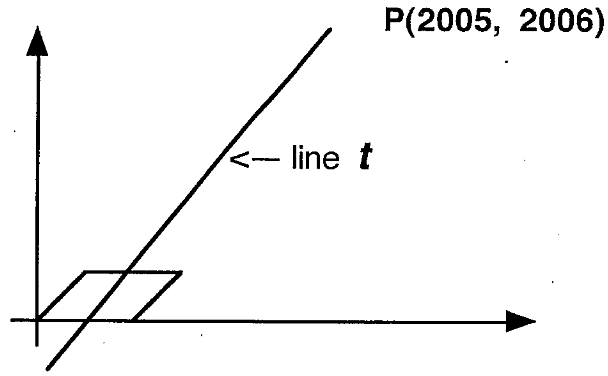
Contest #5 **February 6, 2006**

TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

T–1. In a pie-eating contest, each contestant ate a whole number of pies. The winner ate twice as many pies as the runner-up, 3 times as many as the third-place contestant, and 4 times as many as the person in fourth place. Together, these four contestants ate fewer than 60 pies. What is the greatest number of pies the winner could have eaten?

T–2. A parallelogram has vertices at $(0,0)$, $(2,2)$ $(6,2)$, *and* $(4,0)$. Point **P** has coordinates $(2005, 2006)$. Line t passes through point **P** and divides the parallelogram into two regions of equal area. What is the slope of line t?



T–3. Although four students tried to find the sum of the first 21 positive primes, only one got the correct answer. Pat got 709, Lee got 711, Sandy got 712, and Dale got 713. State the name of the person who got the correct answer.

T–4. To fill **<u>four</u>** big cubes, each with a surface area of 64, I would need n smaller cubes, each with a surface area of 16. What is the value of n?

T–5. What is the ordered pair of positive integers (A, B), with B as small as possible, for which

$$\frac{7}{10} < \frac{A}{B} < \frac{11}{15} ?$$

MCWP Senior High Mathematics League

Contest #6 **INDIVIDUAL QUESTIONS** **March 6, 2006**

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INDIVIDUAL ROUND 1 — 15 MINUTES

1. What per cent of the sum of 20 numbers is the average of these 20 numbers?

2. In a certain rectangle, the length L is a mean proportional between the width W and the semiperimeter, $L+W$. If a square equal in area to this rectangle has side-length kW, what is the value of k?

3. The medians of a triangle are concurrent at a point called the *centroid*. A line through the centroid, parallel to a side of the triangle, splits the triangle into a smaller triangle and a trapezoid. What is the ratio of the area of the smaller triangle to the area of the trapezoid?

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INDIVIDUAL ROUND 2 — 15 MINUTES

4. What is the value of x which satisfies $(\log_7 x)(\log_{11} 7) = 2.5$?

5. How many different (non-congruent) triangles have integer sides and perimeter 15?

6. Determine, in simplest form, the value of x for which

$$\sqrt[3]{54+\sqrt{x}} + \sqrt[3]{54-\sqrt{x}} = \sqrt[3]{18}.$$

MCWP Senior High Mathematics League

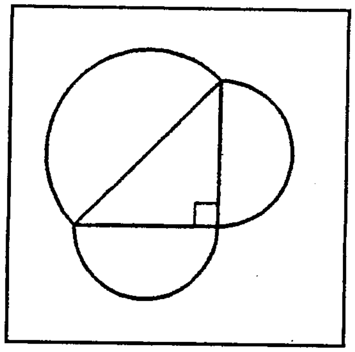
Contest #6 **March 7, 2005**

TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

T–1. Whenever brothers Hocus, Pocus, and Crocus are asked a question, two lie and one tells the truth. I asked them who was youngest. Pocus said Crocus was oldest, Crocus said Hocus was youngest, and Hocus claimed not to be the oldest. Who was the youngest?

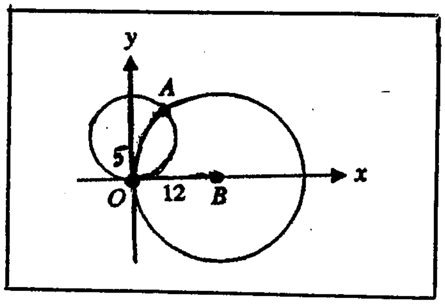
T–2. Each side of an isosceles right triangle is the diameter of a semicircle. What is the area of the triangle if the sum of the areas of the three semicircles is 200π ?



T–3. If $x \neq y$, but $\dfrac{x}{y} + x = \dfrac{y}{x} + y$, what is the value of $\dfrac{1}{x} + \dfrac{1}{y}$?

T–4. What is the value of $b > 0$ for which the region bounded by both the x–axis and $y = -|2x| + b$ has an area of 72?

T–5. Circle $\boldsymbol{A}$ has center (0,5) and radius 5. Circle $\boldsymbol{B}$ has center (12,0) and radius 12. What is the length of a radius of a third circle which passes through the center of circle B and through both points of intersection of circles $\boldsymbol{A}$ and $\boldsymbol{B}$?



SOLUTIONS TO INDIVIDUAL QUESTIONS

Contest #1, 2005-2006 **October 17, 2005**

I–1. If x is the percent of reduction in decimal form, then the price after the first reduction is $25(1-x)$. After the second reduction, the price will be:

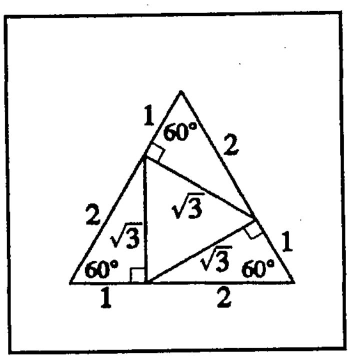
$[25(1-x)](1-x) = 25(1-x)^2 = 16$

Therefore, $(1-x)^2 = \frac{16}{25}$, so that

$1 - x = \frac{4}{5} = 0.80$ and $x = .20 = 20\%$

I–2. In any circle with a chord of length 8, it is always possible to draw a second chord, perpendicular to the first, whose length is any positive number less than 8. Therefore, we want to find the smallest circle in which it is possible to have a chord of length 8. This circle has a diameter of length 8 and an area of 16π.

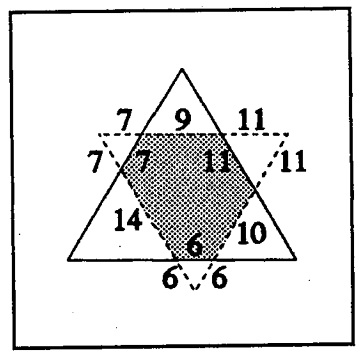
I–3. **METHOD 1:**
Using the diagram, it is easy to see that the perimeter of the smaller equilateral triangle is $3\sqrt{3}$.



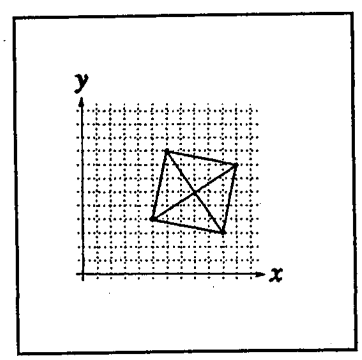
METHOD 2:
Use the law of cosines.

I–4. Let A be Mary's area code.
Since $A + 7$ is divisible by 7, A must be divisible by 7. Similarly, A must also be divisible by 8 and 9. The only 3-digit multiple of 7, 8, and 9 is their least common multiple, 504.

I–5. As can be seen in the diagram, there are two equilateral triangles that can circumscribe this hexagon. The larger triangle has a side-length of 30, and the smaller one has a side-length of 27.



I–6. The given vertices lie on one of the square's diagonals. Since the slope of this diagonal is $-\frac{3}{2}$, the slope of the other diagonal is $\frac{2}{3}$. The midpoint of both diagonals is $(8,6)$. So, the other vertices are at $(5,4)$ *and* $(11,8)$



SOLUTIONS TO TEAM QUESTIONS

Contest #1, 2005 - 2006 **October 17, 2005**

T-1 The "magic sum" is the sum of 7 entries from the magic square (in one row, column or main diagonal). The average entry is 25, so the "magic sum" is 7 X 25 = 175

T-2 By rationalizing the denominator of the left side, we have:

$$\frac{\sqrt{21+x}+\sqrt{21-x}}{\sqrt{21+x}-\sqrt{21-x}} \bullet \frac{\sqrt{21+x}+\sqrt{21-x}}{\sqrt{21+x}+\sqrt{21-x}} = \frac{42+2\sqrt{21^2-x^2}}{21+x-21+x} = \frac{21+\sqrt{21^2-x^2}}{x}$$

This will equal $\frac{21}{x}$ if and only if $\sqrt{21^2-x^2}=0$.

The two solutions, both of which check, are $x = 21,\ -21$

T-3 METHOD 1:

The number of 4-digit numbers that don't contain a "1" is $9 \times 9 \times 9 \times 9$.

This includes 0000, so the number of positive integers less than 10,000 that don't contain a "1" is $9^4 - 1 = 6560$. There are 9999 positive integers less than 10,000.

The number that DO contain one or more "1"s is $9999 - 6560 = 3439$.

METHOD 2:

There are $1 \times 10 \times 10 \times 10 = 1000$ numbers whose first "1" is in the thousands' place.

There are $9 \times 1 \times 10 \times 10 = 900$ numbers whose first "1" is in the hundreds' place.

There are $9 \times 9 \times 1 \times 10 = 810$ numbers whose first "1" is in the tens' place.

There are $9 \times 9 \times 9 \times 1 = 729$ numbers whose first "1" is in the units' place.

The sum of these is 3439.

T-4 To maximize the triangle's area, let 60 be the length of the shortest altitude.

The shortest altitude is drawn to the longest side, the hypotenuse.

Since $(60)(5k) = (3k)(4k)$, we have $k = 25$.

Then the area of the triangle is:

$150k = 150(25) = (100)(25) + (50)(25) = 2500 + 1250 = 3750$

T-5 The sum of the roots of $ax^2 + bx + c = 0$ is $-\frac{b}{a}$, and the product of the roots is $\frac{c}{a}$.

So $r + s = 26$ and $19r + 94s = 1994$.

Multiply the first equation by -19 and add it to the second equation, to get $s = 20$.

Then, $r = 6$, and we have, $rs = \frac{c}{a} = c = 120$

Math Council of Western PA Sr High Math Leauge 2005-2006
SOLUTIONS TO INDIVIDUAL QUESTIONS
Contest #2 — November 14, 2005

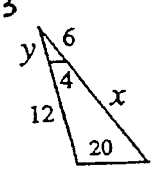
Answers

1.	$\frac{-1+\sqrt{5}}{2}$	3.	21	5.	20
2.	$\frac{12}{11}$	4.	36	6.	400 or \$400

1) Simplifying the fraction, $x=\frac{1}{1+\frac{1}{1+x}}=\frac{1}{\frac{2+x}{1+x}}=\frac{1+x}{2+x}$, so $x^2+x-1=0$.
The only positive solution is $x=\boxed{\frac{-1+\sqrt{5}}{2}}$.

2) They meet the next time shortly after 1 P.M. It takes slightly more than 1 hour from one time the hands overlap to the next such time. They meet 11 times (after noon) up to and including the time they meet at the digit 12, at midnight, 12 hours later. Each of the 11 overlaps comes $\boxed{12/11}$ hours after the previous one.

3



The whole triangle is similar to the upper triangle, so $\frac{20}{4}=\frac{5}{1}=\frac{x+6}{6}=\frac{y+12}{y}$. Solving, $x=24, y=3$, $x-y=\boxed{21}$.

4) Method I: Work backwards. Mo had 1 coin. How many did Miney find? She found twice as many coins as were left before she took the extra 2. So she found $2(1+2)=6$ coins. Similarly, Meanie left 6, so began with $2(6+2)=16$. Finally, Eanie found $2(16+2)=\boxed{36}$ coins at the outset.
Method II $\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}x-2\right)-2\right)-2=1$, so $x=36$.

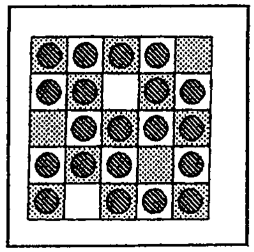
5 $x^2+x-5=0 \iff x^2+x=5 \Rightarrow (x^2+x)^2=5^2 \Rightarrow x^4+2x^3+x^2=25$,
so $x^4+2x^3+x^2-5=25-5=\boxed{20}$.

6) $2(10S+4G+P=1690)$ $\quad 21S+9G+3P=3780$
$3(7S+3G+P=1260)$ $\quad 20S+8G+2P=3380$
Subtracting $\quad S+G+P=\boxed{400}$.

Math Council of Western PA Sr. High Math League 2005 - 2006

Contest # 2 — SOLUTIONS TO TEAM ROUND November 14, 2005

T–1. You cannot use more than 4 checkers in each row, so the number of checkers used is at most 20. The diagram shows one of several possible ways to use the maximum number of checkers, 20.



T–2. To minimize the number of 17¢ items, we need to maximize the number of 23¢ items. So, divide $10.00 = 1000¢ by 23¢, and the remainder is 11¢, which isn't divisible by 17¢. With one fewer 23¢ item, the remainder would be 11¢ + 23¢ = 34¢, which is divisible by 17¢, giving a quotient of 2, which is the required solution.

T–3. Since 20 books cost $260, and 16 books cost $256, it's cheaper to buy 20 books than it is to buy n books, if n = **17, 18, or 19.**

T–4. $\log_2\left(\frac{x-1}{x+1}\right) > 1 \Rightarrow \frac{x-1}{x+1} > 2 \Rightarrow \frac{x-1}{x+1} - 2 > 0$. Continuing to solve, we have $\frac{-x-3}{x+1} > 0 \Rightarrow \frac{x+3}{x+1} < 0$. This inequality is true if and only if $(x+3)$ *and* $(x+1)$ have opposite signs. If $x > -1$, both are positive; and if $x < -3$, both are negative. The solution, therefore, is $-3 < x < -1$.

T–5. **METHOD 1:**

In this progression, with first term a and common difference d, the sum of the third and fifth terms is $a + 2d + a + 4d = 14$. Simplifying, we have $a + 3d = 7$.

Now, adding the first 12 terms, we have

$a + (a+d) + (a+2d) + \cdots + (a+11d) = 12a + 66d = 129$.

Multiply the first equation by 12 and subtract from the second equation:

$$\begin{array}{r} 12a + 66d = 129 \\ \underline{12a + 36d = \ \ 84} \end{array}$$

$$30d = 45 \Rightarrow d = \frac{3}{2} \Rightarrow a = \frac{5}{2}$$

Finally, if $193 = a + (n-1)d$, then it follows that $n = 128$.

METHOD 2:

Since $a + 3d$ is the fourth term, and since the value of this term is 7, the first 12 terms are: $7 - 3d,\ 7 - 2d,\ 7 - d,\ 7,\ 7 + d, \ldots, 7 + 8d$. Their sum is $84 + 30d = 129$, so $d = \frac{3}{2}$ *and* $a = \frac{5}{2}$. Then, if $193 = a + (n-1)d$, it follows that $n = 128$

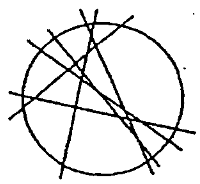
Math Council of Western PA Sr High Math Leauge 2005 - 2006
SOLUTIONS TO INDIVIDUAL QUESTIONS
Contest #3 — December 12, 2005

Answers

1. \$33.30
2. 22
3. \$1.90
4. 41
5. $\{x \mid x < 0\}$ (set notation not required)
6. $\sqrt{18}$

1) $.05 + .10 + .15 + \ldots + \underbrace{36(.05)}_{1.80} = \frac{36}{2}(.05 + 1.80) = 18(1.85) = \boxed{\$33.30}$.

2) The circle itself is 1 piece. When the first cut is made, one more piece is added to make 2 pieces in all. The 2nd cut adds 2 more pieces, making 4 in all. In general, "cut n" adds n more pieces, as long as the new cut crosses each of the previous cuts in new intersection points. For 6 cuts, we have $2+2+3+4+5+6 = \boxed{22}$ pieces

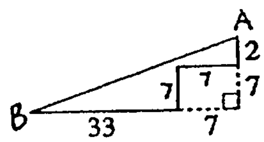


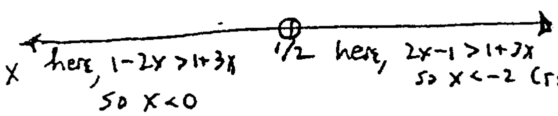
3) $\left.\begin{array}{l} n + 2 = 3d \\ 2(d+3) = n \end{array}\right\}$ solving, $n = 22$, $d = 8$. total value $= \boxed{\$1.90}$.

4) $9^2 + 40^2 = AB^2$

$1681 = AB^2$

$AB = \boxed{41}$



5) 

Thus, $\boxed{x < 0}$.

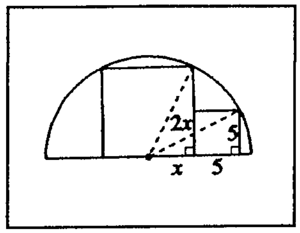
6) $x^2 + xy = 7 \Rightarrow x(x+y) = 7$

$y^2 + xy = 11 \Rightarrow y(x+y) = 11$

$x(x+y) + y(x+y) = 18 \Leftrightarrow (x+y)^2 = 18$, so, since $x > 0$ and $y > 0$, $x + y = \boxed{\sqrt{18}}$.

Math Council of Western PA Sr. High Math League 2005 - 2006

Contest # 3 — SOLUTIONS TO TEAM ROUND **December 12, 2005**



T–1. In the diagram, each dotted segment is both the hypotenuse of a right triangle and a radius of the circle; so the dotted segments are congruent. In one right triangle, the legs are x and 2x. In the other, the legs are 5 and x+5. By the Pythagorean Theorem, $x^2 + (2x)^2 = 5^2 + (x+5)^2$. Solving, we have $x = 5$.
So, a side of the large square is 10 and the area of the large square is 100.

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T–2. Brian has a total of 88¢, so this is the largest possible sum he can make. The question remains whether he can make all the values between 1¢ and 88¢. With only 3 pennies, he cannot make the amounts ending in 4 or 9 -- 4¢, 9¢, 14¢, 19¢, 24¢, 29¢, 34¢, 39¢, 44¢, 49¢, 54¢, 59¢, 64¢, 69¢, 74¢, 79¢, and 84¢. Therefore, with the coins he has, he can make 88 – 17 = 71 different sums.

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T–3. When x is divided by y, the quotient is 3 and the remainder is 7, so $x = 3y + 7$. Divide this by 2y to get $\frac{x}{2y} = \frac{3y}{2y} + \frac{7}{2y} = \frac{2y}{2y} + \frac{y}{2y} + \frac{7}{2y} = 1 + \frac{y+7}{2y}$. When we divide x by y, the remainder is 7, so $y > 7$. Since y is integral, $y \geq 8$, making $0 < \frac{y+7}{2y} < 1$. Therefore, the quotient, q, is 1 and the remainder, r, is $y + 7$. Since $y \geq 8$, the least possible value of r is 15.

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T–4. Rewriting, we have $3^{2\sqrt{3x}} + 3 = 4\left(3^{\sqrt{3x}}\right)$. If we let $y = 3^{\sqrt{3x}}$, then the equation becomes, $y^2 - 4y + 3 = 0$. Solving this, we have $y = 3$ *or* $y = 1$. Then $3^{\sqrt{3x}} = 3 \Rightarrow x = \frac{1}{3}$ *or* $3^{\sqrt{3x}} = 1 \Rightarrow x = 0$. So, the solutions are $\frac{1}{3}$ *and* 0

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T–5. When N is even and divisible by 4, then N^2 will be divisible by 16 (and, therefore by 8 as well). When N is even but not divisible by 4, then N^2 will be divisible by 4, but not by 8. Of the 1002 even integers from 1 to 2005, half are divisible by 4, and therefore divisible by 8 as well. The required probability is $\frac{1}{2}$

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Math Council of Western PA Sr High Math Leauge 2005 - 2006
SOLUTIONS TO INDIVIDUAL QUESTIONS
Contest #4 — January 9, 2006

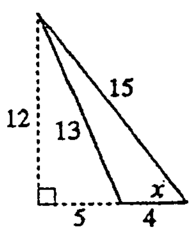
Answers

1. 216
2. 178 or 178°
3. $\frac{4}{3}$ or $1.\overline{3}$ or 1.333
4. 3
5. 74
6. $\sqrt{6}$

1) Since the dividend is 6 times the divisor, the quotient must be 6. The divisor is 6 times the quotient, or 36. The dividend is $6 \times 36 = \boxed{216}$.

2) The measures of the angles of the pentagon are: 38, 38+d, 38+2d, 38+3d, 38+4d. In a pentagon, the sum of the interior angles is 540°, so, adding, $190 + 10d = 540$, $10d = 350$, so $d = 35$ and $38 + 4d = \boxed{178}$.

3)



$\tan x = \frac{12}{9} = \boxed{\frac{4}{3}}$

Alternatively, use the law of cosines to get $\cos x = \frac{3}{5}$ and use that to determine the value of $\tan x$.

4) If both exponents are 0, $2x^2 - 7x + 3 = x^2 - x - 6$

$x^2 - 6x + 9 = 0$

$(x-3)^2 = 0$

$x = \boxed{3}$. There is another solution, but it is irrational.

Check

Note: At $x = 3$, $2x^2 - 7x + 3 = 0$, and $x^2 - x - 6 = 0$ as well.

5) $f(x) = x^2 + 7x + k$, so $f(k) = -16 = k^2 + 7k + k \Leftrightarrow k^2 + 8k + 16 = 0$

$(k+4)^2 = 0$

$k = -4$

$f(6) = 6^2 + 7(6) - 4 = 36 + 42 - 4 = \boxed{74}$.

6) Let $x = \sqrt{5+\sqrt{21}} - \sqrt{5-\sqrt{21}}$. Squaring,

$x^2 = 5 + \sqrt{21} + 5 - \sqrt{21} - 2\sqrt{(5+\sqrt{21})(5-\sqrt{21})}$

$= 10 - 2\sqrt{25-21} = 10 - 2(2)$

$= 6$, so $x = \boxed{\sqrt{6}}$, since $x > 0$.

Math Council of Western PA Sr High Math Leauge 2005 - 2006
SOLUTIONS TO INDIVIDUAL QUESTIONS
Contest #5 — February 6, 2006

Answers

1. $4\sqrt{2}$ or $\sqrt{32}$
2. $3y$
3. $\frac{5}{4}$ or 1.25 or exact equivalent
4. 12
5. (42,41), (50,51) Both required
6. $\frac{7}{24}$

1) $s^2 = 4s \Rightarrow s = 4 \Rightarrow d = \boxed{4\sqrt{2}}$ or $\sqrt{32}$.

2) $(f(a))^3 = y^3$; $f(a^3) = a^3 + \frac{1}{a^3}$. $y = a + \frac{1}{a}$, so $y^3 = (a+\frac{1}{a})^3 = a^3 + \frac{1}{a^3} + 3(a+\frac{1}{a})$

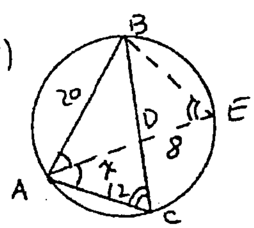
$y^3 = a^3 + \frac{1}{a^3} + 3y$

$y^3 - f(a^3) = a^3 + \frac{1}{a^3} + 3y - (a^3 + \frac{1}{a^3}) = \boxed{3y}$.

3) $(2x)^{2006} = 5^{1003}x^{1003} \Leftrightarrow 2^{2006}x^{2006} = 5^{1003}x^{1003}$. Divide by x^{1003}.

$4^{1003}x^{1003} = 5^{1003}$

$x = \boxed{\frac{5}{4}}$.

4) 

$\triangle ACD \sim \triangle AEB$, so $\frac{x}{20} = \frac{12}{x+8} \Rightarrow x^2 + 8x - 240 = 0$

$(x-12)(x+20) = 0$ reject

$x = \boxed{12}$.

5) either $5a - 4(a+1) = 46 \Rightarrow a = 50$ $\quad (x,y) = \boxed{(50,51) \text{ or } (42,41)}$

or $5(a+1) - 4(a) = 46 \Rightarrow a = 41$ $\quad$ Both required

6) $S_1 = \frac{1}{5} + \frac{1}{125} + \dots = \frac{\frac{1}{5}}{1-\frac{1}{25}} = \frac{5}{24}$

$S_2 = \frac{2}{5}S_1 = \frac{2}{24}$ $\quad S = S_1 + S_2 = \boxed{\frac{7}{24}}$.

Math Council of Western PA Sr. High Math League 2005 - 2006

Contest # 5 — SOLUTIONS TO TEAM ROUND February 6, 2006

T–1. If the winner ate x pies, then the runner-up ate x/2 pies, the 3rd-place contestant ate x/3 pies, and the 4th-place contestant ate x/4 pies. The total number of pies eaten was 25x/12. To make both x and 25x/12 integral, x must be a multiple of 12. Since $\frac{25x}{12} < 60$, x = 12 or 24, so the maximum value of x is 24.

T–2. METHOD 1: Since every parallelogram is symmetric about the intersection of its diagonals, any line which divides the parallelogram into two regions of equal area must pass through this intersection point. This point is (3,1), the midpoint of the diagonals. Since line t passes through (3, 1) and (2005, 2006), its slope is 2005/2002.

METHOD 2: The parallelogram has vertices A(0, 0), B(4, 0), C(6, 2), and D(2, 2). There must exist a point E(x, 0) where the line t crosses the x-axis. For the parallelogram to be split equally, the distance from A(0, 0) to E(x, 0) must equal the distance from C(6, 2) to F(6-x, 2). Since E, F and P are collinear points, we can solve for x.

T–3. The sum of the one even number (i.e. 2) and the twenty odd numbers (all the others) is always even, so Sandy had the correct answer.

T–4. METHOD 1: Since the ratio of the surface areas of the cubes is $4:1 = 2^2:1^2$, the ratio of their edges is 2:1, and the ratio of their volumes is $2^3:1^3 = 8:1$. Therefore it will take 8 smaller cubes to fill each larger one, and the value of n is $4 \times 8 = 32$.

METHOD 2: For one large cube, $6S^2 = 64 \Rightarrow S = \frac{8}{\sqrt{6}} \Rightarrow V = \frac{8^3}{6\sqrt{6}}$

For one smaller cube, $6s^2 = 16 \Rightarrow s = \frac{4}{\sqrt{6}} \Rightarrow v = \frac{4^3}{6\sqrt{6}}$

Dividing the volumes, we have, $\frac{V}{v} = \frac{8^3}{4^3} = \frac{2^3 4^3}{4^3} = 2^3 = 8$

So, it will take $4 \times 8 = 32$ smaller boxes and $n = 32$

T–5. In $\frac{7}{10} < \frac{A}{B} < \frac{11}{15}$, both A and B are positive integers, so we can multiply through by $30B$ to get the equivalent inequality, $21B < 30A < 22B$. If $B = 1$, A will not be integral. The first value of B for which there is a multiple of 30 between 21B and 22B is B = 7. In that case,
$147 < 150 < 154$, so $(A, B) = (5, 7)$

Math Council of Western PA Sr High Math Leauge 2005 - 2006
SOLUTIONS TO INDIVIDUAL QUESTIONS
Contest #6 — March 6, 2006

Answers

1. 5 or 5%
2. $\sqrt{\frac{1+\sqrt{5}}{2}}$ or exact equivalent
3. 4:5 or $\frac{4}{5}$ or 0.8
4. $121\sqrt{11}$ or $11^{5/2}$
5. 7
6. 4416

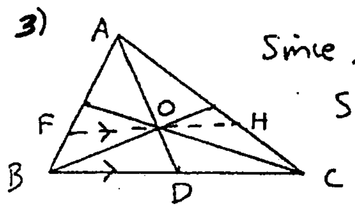
1) Let the average be n. Then the sum is $20n$. $\frac{n}{20n} = 0.05 = \boxed{5\%}$.

2) $\frac{L+W}{L} = \frac{L}{W} \Leftrightarrow L^2 - WL - W^2 = 0$. By the quadratic formula,

Note: Alternatively, let W=1 and solve for L.

$L = \frac{W \pm \sqrt{W^2+4W^2}}{2} = W\left(\frac{1+\sqrt{5}}{2}\right)$. The area $= W^2\left(\frac{1+\sqrt{5}}{2}\right)$
$(L>0)$ The side of the square $= W\boxed{\sqrt{\frac{1+\sqrt{5}}{2}}}$.

3)



Since $\frac{AO}{OD} = \frac{2}{1}$, $\frac{AF}{FB} = \frac{2}{1}$ $\triangle AFH \sim \triangle ABC$ with ratio of similitude of $\frac{2}{2+1} = \frac{2}{3}$. The ratio of their areas is $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$, so $\triangle AFH$ is $\frac{4}{9}$ of $\triangle ABC$, and the trapezoid occupies the remaining $\frac{5}{9}$. Ratio $= \frac{4/9}{5/9} = \boxed{\frac{4}{5}}$.

4) Change of Base theorem: $\frac{\log x}{\cancel{\log 7}} \cdot \frac{\cancel{\log 7}}{\log 11} = \frac{5}{2} \Leftrightarrow \log x = \log 11^{5/2}$

so $x = \boxed{121\sqrt{11}}$.

5) $(1,7,7), (2,6,7), (3,6,6), (3,5,7), (4,5,6), (4,4,7), (5,5,5)$

a total of $\boxed{7}$.

6) Cubing both sides, $54+\sqrt{x} + 3\sqrt[3]{54^2-x}\underbrace{\left(\sqrt[3]{54+\sqrt{x}} + \sqrt[3]{54-\sqrt{x}}\right)}_{=\sqrt[3]{18}} + 54 - \sqrt{x} = 18$

Thus, $3\sqrt[3]{18(54^2-x)} = -90$, or, dividing by 3
and cubing $18(54^2-x) = -27000$
$54^2 - x = -1500$ $[54^2 = 2916]$
$-x = -4416$, so $x = \boxed{4416}$.

Math Council of Western PA Sr. High Math League 2005 - 2006

Contest # 6 — SOLUTIONS TO TEAM ROUND **March 6, 2006**

T–1. **METHOD 1:**
If Hocus lied, he'd be the oldest, but all 3 would then be liars — so Hocus told the truth and the others lied. From Crocus' lie and Hocus' statement, Hocus is not the youngest or the oldest. From Pocus' lie, Crocus isn't oldest, so the youngest is **Crocus**.

METHOD 2:
From youngest to oldest, the possibilities are: 1)DHP, 2)CPH, 3)HCP, 4)HPC, 5)PCH, 6)PHC. Pocus is truthful in only 4 and 6. Crocus is truthful in only 3 and 4. Hocus is truthful in only 1, 3, 4 and 6. The only possibility consistent with all requirements is 1)CHP. So **Crocus** is the youngest.

T–2. If the length of each leg of the isosceles triangle is 2r, then the length of the hypotenuse is $2r\sqrt{2}$. The sum of the areas of the three semicircles is $\frac{1}{2}\pi r^2 + \frac{1}{2}\pi r^2 + \frac{1}{2}\pi\left(r\sqrt{2}\right)^2 = 2\pi r^2$. Hence, $2\pi r^2 = 200\pi$. The area of the isosceles right triangle is $2r^2 = 200$

T–3. $\frac{x}{y} + x = \frac{y}{x} + y \Rightarrow x^2 + x^2y = y^2 + xy^2 \Rightarrow x^2 - y^2 = -x^2y + xy^2$
$\Rightarrow (x+y)(x-y) = -xy(x-y)$. Since $x \neq y$, *then* $(x-y) \neq 0$,
so $x + y = -xy$ and $\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} = \frac{-xy}{xy} = -1$

T–4. Since $b > 0$, our concern is with the graph as it appears in quadrants I and II. In quadrant I, since $x > 0$, $y = -2x + b$. In quadrant II, since $x < 0$, we have $y = -2(-x) + b = 2x + b$. Since these lines have y-intercepts of b, and since each has $|slope| = 2$, they each cross the y-axis b units above the origin and cross the x-axis $\frac{1}{2}b$ units from the origin. The result is an isosceles triangle with an altitude of length b (on the y-axis) and a base of length b (on the x-axis). The area of this triangle is $\frac{1}{2}(b)(b) = \frac{1}{2}b^2 = 72$. Therefore, b = 12.

T–5. The perpendicular bisector of $\overline{OB}$ is x = 6. The perpendicular bisector of $\overline{AO}$ is line DB. Its equation is $y = -\frac{5}{12}x + 5$. The point $\left(6, \frac{5}{2}\right)$, at which $\overline{AO}$ intersects $\overline{DB}$ is the center of the circumscribed circle. The distance from $\left(6, \frac{5}{2}\right)$ to $(0,0)$ is 13/2, the radius of the circumscribed circle.